THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 1: Preliminaries

1. Show that if $f : A \to B$ and E, F are subsets of A, then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$.

2. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

3. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.
   (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.

4. (a) Suppose that f is an injection. Show that $f^{-1} \circ f(x) = x$ for all $x \in D(f)$ and that $f \circ f^{-1}(y) = y$ for all $y \in R(f)$.
   (b) If f is a bijection of A onto B, show that f^{-1} is a bijection of B onto A.

5. Prove that if $f : A \to B$ is bijective and $g : B \to C$ is bijective, then the composite $g \circ f$ is a bijective map of A onto C.

6. Let $f : A \to B$ and $g : B \to C$ be functions.
   (a) Show that if $g \circ f$ is injective, then f is injective.
   (b) Show that if $g \circ f$ is surjective, then g is surjective.

7. Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

8. Prove that $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$ for all $n \in \mathbb{N}$

9. Prove the *second version* of Principle of Mathematical Induction:
   Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a statement for each natural number $n \geq n_0$. Suppose that
   - The statement $P(n_0)$ is true.
   - For all $k \geq n_0$, the truth of $P(k)$ implies the truth of $P(k+1)$.

   Then $P(n)$ is true for all $n \geq n_0$.

10. Prove the *strong version* of Principle of Mathematical Induction:
    Let S be a subset of $\mathbb{N}$ such that
    - $1 \in S$.
    - For every $k \in \mathbb{N}$, if $\{1, 2, \cdots, k\} \subseteq S$, then $k + 1 \in S$.

    Then $S = \mathbb{N}$.

11. Prove a variation of Principle of Mathematical Induction:
    Let S be a subset of $\mathbb{N}$ such that

    - $2^k \in S$ for all $k \in \mathbb{N}$.
    - If $k \in S$ and $k \geq 2$, then $k - 1 \in S$.

    Then $S = \mathbb{N}$.

12. Show that the set $S = \{n \in \mathbb{N} : n \geq 2015\}$ is countably infinite.

13. Prove that if S and T are countably infinite, then $S \cup T$ is countably infinite.

14. Prove that if S is countably infinite and T is finite, then S/T is countably infinite.

15. Suppose that $f : S \to T$ is an injective function, where S is an infinite set. Prove that T is an infinite set.

16. Suppose that $f : S \to T$ is an surjective function, where T is a countably infinite set. Is S an infinite set? Why?

17. Let S be a set. $\mathcal{P}(S)$ is defined to be the collection of all subsets of S.

    (a) Write down $\mathcal{P}(S)$ explicitly if $S = \{1, 2, 3\}$. How many elements does $\mathcal{P}(S)$ contain?

    (b) Use mathematical induction to prove that if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.